

On conjectures by Csordas, Charalambides and Waleffe*

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In the present note we obtain new results on two conjectures by Csordas et al. regarding the interlacing property of zeros of special polynomials. These polynomials came from the Jacobi tau methods for the Sturm-Liouville eigenvalue problem. Their coefficients are the successive even derivatives of the Jacobi polynomials $P_n^{(\alpha, \beta)}$ evaluated at the point one. The first conjecture states that the polynomials constructed from $P_n^{(\alpha, \beta)}$ and $P_{n-1}^{(\alpha, \beta)}$ are interlacing when $-1 < \alpha < 1$ and $-1 < \beta$. We prove it in a range of parameters wider than that given earlier by Charalambides and Waleffe. We also show that within narrower bounds another conjecture holds. It asserts that the polynomials constructed from $P_n^{(\alpha, \beta)}$ and $P_{n-2}^{(\alpha, \beta)}$ are also interlacing.

Keywords: Jacobi polynomials · Interlacing zeros · Tau methods · Hurwitz stability · Hermite-Biehler theorem

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1 Introduction

This study is devoted to properties of zeros of polynomials which originate from mathematical physics. The orthogonal polynomials proved to be a very helpful tool for the discretization of linear differential operators. The main feature of the tau methods is the adoption of a polynomial basis which does not automatically satisfy the boundary conditions. This induces a problem at the boundary (i.e. the points ± 1 in the Jacobi case). The family of polynomials studied here is connected this way to the eigenproblem $u''(x) = \lambda u(x)$ on the interval $x \in (-1, 1)$ with various homogeneous boundary conditions (for the details see [4, 3]). We place our main emphasis on the analytic properties of the considered family itself, leaving aside the corresponding properties of the original differential operators. More information on the tau methods can be found in e.g. [1, §10.4.2].

The *Jacobi polynomials* (see their definition and basic properties in e.g. [7, Ch. IV])

$$P_n^{(\alpha, \beta)}(x) = \binom{n + \alpha}{n} {}_2F_1 \left[\begin{matrix} -n, & n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right], \quad n = 1, 2, \dots$$

appear regularly in applications as classical orthogonal polynomials. They are more general than those of Chebyshev, Legendre and Gegenbauer. The Jacobi polynomials are orthogonal with respect to the measure $w_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta$ on the interval $(-1, 1)$ whenever both the parameters α and β are greater than -1 :

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) w_{\alpha, \beta}(x) dx = 0 \quad \text{if } k \neq n.$$

The usual normalization supposes that $P_n^{(\alpha, \beta)}(1) = \binom{n + \alpha}{n} = \frac{(\alpha + 1)_n}{n!}$, where we applied the so-called *Pochhammer symbol* or the *rising factorial* defined as

$$(\alpha + 1)_n := (\alpha + 1) \cdot (\alpha + 2) \cdots (\alpha + n).$$

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In this notation we have

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k! (\alpha+1)_k} \left(\frac{1-x}{2}\right)^k, \quad n = 1, 2, \dots \quad (1)$$

Definition (see [2, p. 17]). We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace* (or *interlace strictly*) if the following conditions hold simultaneously:

- all zeros of $g(x)$ and $h(x)$ are simple, real and distinct (i.e. the polynomials are coprime),
- between each two consecutive zeros of $g(x)$ there is exactly one zero of the polynomial $h(x)$, and
- between each two consecutive zeros of $h(x)$ there is exactly one zero of the polynomial $g(x)$.

We say that the zeros of the polynomials $g(x)$ and $h(x)$ *interlace non-strictly* if their zeros are real and become strictly interlacing after dividing both polynomials by the greatest common divisor $\gcd(g, h)$. Roughly speaking, the zeros of two polynomials interlace non-strictly if they can meet but never pass through each other when changing continuously from a strictly interlacing state.

Definition. A pair $(g(x), h(x))$ is called *real* if for any real numbers A, B the combination $Ag(x) + Bh(x)$ has only real zeros. This is equivalent to the non-strict interlacing property of $g(x)$ and $h(x)$, which is shown in e.g. [2, Chapter I].

Remark. The phrases “ $g(x)$ and $h(x)$ interlace”, “ $g(x)$ and $h(x)$ possess the interlacing property”, “ $g(x)$ interlaces $h(x)$ ”, “ $g(x)$ and $h(x)$ have interlacing zeros” and “the zeros of $g(x)$ and $h(x)$ are interlacing” we use synonymously.

It is well-known that the orthogonal polynomials on the real line have real interlacing zeros (due to the so-called three-term recurrence; see e.g. [7, pp. 42–47, Sections 3.2–3.3]). That is, in particular, the zeros of $P_n^{(\alpha, \beta)}$ and $P_{n-1}^{(\alpha, \beta)}$ interlace for all natural n . In the present note we study zeros of polynomials that do not satisfy the three-term recurrence. More specifically, we consider

$$\phi_n^{(\alpha, \beta)}(\mu) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha, \beta)}(x) \Big|_{x=1} \cdot \mu^k = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-n)_{2k} (n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}} \left(\frac{\mu}{4}\right)^k, \quad n = 1, 2, \dots, \quad (2)$$

where the notation $[a]$ stands for the integer part of the number a .

Theorem CCW (Csordas, Charalambides and Waleffe [4]). *For every positive integer $n \geq 2$ the polynomial $\phi_n^{(\alpha, \beta)}(\mu)$, $-1 < \alpha < 1$, $-1 < \beta$, has only real negative zeros.*

The proof of this theorem given in [4] rests on the Hermite-Biehler theory (see Theorem HB herein).

Remark (to Theorem CCW). In fact, the authors have shown that $\phi_n^{(\alpha, \beta)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha+1, \beta+1)}(\mu)$. As a result, the theorem remains valid when $1 \leq \alpha < 2$ and $0 < \beta$. Note that the interlacing property here is strict, so it implies the simplicity of the zeros. Furthermore, the polynomial $\phi_n^{(\alpha, \beta)}(\mu)$ has only simple negative zeros for $-1 < \alpha < 0$ and $-2 < \beta$, as well. This follows as a straightforward consequence of Theorems CW* and Lemma 2 of the present study.

Based on Theorem CCW the authors of [4] conjectured that these polynomials also have the following property.

Conjecture A ([4, p. 3559]). *For $-1 < \alpha < 1$, $-1 < \beta$ and $n \geq 4$ the zeros of the polynomials $\phi_n^{(\alpha, \beta)}$ and $\phi_{n-1}^{(\alpha, \beta)}$ interlace.*

In particular, this assertion would imply that the spectra of polynomial approximations to the corresponding differential operator are negative and simple (see [3]). In [3], Conjecture A was proved for $-1 < \alpha, \beta < 0$ and $0 < \alpha, \beta < 1$: see Theorem CW below. In fact, the upper bound on β is redundant. Theorem CW* with a shorter proof states that the conjecture holds true for

$$-1 < \alpha < 0, -1 < \beta \quad \text{or} \quad 0 \leq \alpha < 1, 0 < \beta \quad \text{or} \quad 1 \leq \alpha < 2, 1 < \beta. \quad (3)$$

Additionally, we study another assertion about the same polynomials.

Conjecture B ([4, p. 3559]). *For $\phi_n^{(\alpha, \beta)}(\mu)$ as in Theorem CCW and for all $n \geq 5$, the zeros of the polynomials $\phi_n^{(\alpha, \beta)}(\mu)$ and $\phi_{n-2}^{(\alpha, \beta)}(\mu)$ interlace.*

Originally, this conjecture was stated for $-1 < \alpha < 1$ and $\beta > -1$. However, numerical calculations show

that it fails for some values satisfying $-1 < \beta < 0 < \alpha < 1$. Our (partial) solution to Conjecture B is given in Theorem 10: it holds true for $-1 < \alpha < 0 < \beta$ or $0 < \alpha < 1 < \beta$. We approach by extending the idea of [3] to another pair of auxiliary polynomials. Certainly, there exists a relation between Conjecture A and Conjecture B as discussed in Section 5.

Vieta's formulae imply that the sum of all zeros of $\phi_n^{(\alpha, \beta)}(\mu)$ tends to $-1/2$ for even n and to $-1/6$ for odd n as $n \rightarrow \infty$. Thus, the assertion of Conjecture B gives that the zero points of $\phi_n^{(\alpha, \beta)}(\mu)$ converge monotonically in n outside of any fixed interval containing the origin. If the assertions of both conjectures hold, then the fraction $\phi_{2n-1}^{(\alpha, \beta)}(\mu) / \phi_{2n}^{(\alpha, \beta)}(\mu)$ maps the upper half of the complex plane into itself and converges to a function meromorphic outside of any disk centred at the origin. This situation resembles how the quotients of orthogonal polynomials of the first and second kinds behave.

Section 2 of the present paper introduces connections between polynomials $\phi_n^{(\alpha, \beta)}(\mu)$ with different n , α and β . These connections allow us to extend and clarify the result [3] in Section 3 (see Theorem CW*). We show that Conjecture A holds true under the conditions (3). Section 4 contains the proof of Conjecture B for $-1 < \alpha < 0 < \beta$ and $0 < \alpha < 1 < \beta$ (see Theorem 10). In the last section we show that the studied conjectures are actually related.

2 Basic relations between the polynomials $\phi_n^{(\alpha, \beta)}$ for various α and β

Being connected with the Jacobi polynomials, the family $\left(\phi_n^{(\alpha, \beta)}\right)_n$, where $n = 2, 3, \dots$, inherits some of their properties. The formulae induced by the corresponding relations for the Jacobi case include (we omit the argument μ of $\phi_n^{(\alpha, \beta)}$ for brevity's sake):

$$(2n + \alpha + \beta)\phi_n^{(\alpha, \beta-1)} = (n + \alpha + \beta)\phi_n^{(\alpha, \beta)} + (n + \alpha)\phi_{n-1}^{(\alpha, \beta)}, \quad (4)$$

$$(2n + \alpha + \beta)\phi_n^{(\alpha-1, \beta)} = (n + \alpha + \beta)\phi_n^{(\alpha, \beta)} - (n + \beta)\phi_{n-1}^{(\alpha, \beta)}, \quad (5)$$

$$(n + \alpha + \beta)\phi_n^{(\alpha, \beta)} \stackrel{(4)+(5)}{=} (n + \beta)\phi_n^{(\alpha, \beta-1)} + (n + \alpha)\phi_n^{(\alpha-1, \beta)}, \quad (6)$$

$$\phi_{n-1}^{(\alpha, \beta)} \stackrel{(4)-(5)}{=} \phi_n^{(\alpha, \beta-1)} - \phi_n^{(\alpha-1, \beta)}. \quad (7)$$

The latter two identities contain labels of equations above the equality sign: we use the convention that this explains how the equalities can be obtained (up to some proper coefficients). For example, $2n + \alpha + \beta$ times (6) is the sum of $n + \beta$ times (4) and $n + \alpha$ times (5), whereas $2n + \alpha + \beta$ times (7) is the difference (4) – (5). The identities (4) and (5) can be checked by applying the formulae (see, e.g., [6, p. 737])

$$(2n + \alpha + \beta)P_n^{(\alpha, \beta-1)}(x) = (n + \alpha + \beta)P_n^{(\alpha, \beta)}(x) + (n + \alpha)P_{n-1}^{(\alpha, \beta)}(x), \quad (8)$$

$$(2n + \alpha + \beta)P_n^{(\alpha-1, \beta)}(x) = (n + \alpha + \beta)P_n^{(\alpha, \beta)}(x) - (n + \beta)P_{n-1}^{(\alpha, \beta)}(x), \quad (9)$$

respectively, to the left-hand side of the definition (2). The relations involving derivatives (not surprisingly) differ from those for the Jacobi polynomials:

$$(n + \alpha)\phi_{n-1}^{(\alpha, \beta+1)} = n\phi_n^{(\alpha, \beta)} - 2\mu\left(\phi_n^{(\alpha, \beta)}\right)', \quad (10)$$

$$(n + \alpha + \beta + 1)\phi_n^{(\alpha, \beta+1)} \stackrel{(10) \text{ and } (4)}{=} (n + \alpha + \beta + 1)\phi_n^{(\alpha, \beta)} + 2\mu\left(\phi_n^{(\alpha, \beta)}\right)', \quad (11)$$

$$(n + \alpha)\phi_n^{(\alpha-1, \beta+1)} \stackrel{(5), \text{ then } (11)-(10)}{=} \alpha\phi_n^{(\alpha, \beta)} + 2\mu\left(\phi_n^{(\alpha, \beta)}\right)', \quad (12)$$

$$\frac{n + \beta}{n + \alpha + \beta}2\mu\left(\phi_n^{(\alpha, \beta-1)}\right)' \stackrel{(11) \text{ and } (6)}{=} (n + \alpha)\phi_n^{(\alpha-1, \beta)} - \alpha\phi_n^{(\alpha, \beta)}, \quad (13)$$

$$2\mu\left(\phi_n^{(\alpha, \beta-1)}\right)' \stackrel{(10) \text{ and } (7)}{=} n\phi_n^{(\alpha-1, \beta)} - \alpha\phi_{n-1}^{(\alpha, \beta)}. \quad (14)$$

So we see that the presented identities are not independent in the sense that they can be obtained as combinations of others with various α and β . We derive the formula (10) with the help of the right-hand side

of (2).

$$\begin{aligned}
 n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-n)_{2k}(n+\alpha+\beta+1)_{2k}}{(\alpha+1)_{2k}} \left(\frac{\mu}{4}\right)^k (n-2k) \\
 &= (n+\alpha) \frac{(\alpha+1)_{n-1}}{(n-1)!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-n+2k) \frac{(-n)(-n+1)\cdots(-n+2k-1)(n+\alpha+\beta+1)_{2k}}{(-n)(\alpha+1)_{2k}} \left(\frac{\mu}{4}\right)^k \\
 &= (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)}.
 \end{aligned}$$

Certainly, we can combine formulae further obtaining e.g.

$$n\phi_n^{(\alpha,\beta)} - 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' \stackrel{(10)}{=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta+1)} \stackrel{(11)}{=} (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} + \frac{n+\alpha}{n+\alpha+\beta} 2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)', \quad (15)$$

$$n\phi_n^{(\alpha,\beta)} - (n+\alpha)\phi_{n-1}^{(\alpha,\beta)} \stackrel{(11)-(10)}{=} \frac{2n+\alpha+\beta}{n+\alpha+\beta} 2\mu\left(\phi_n^{(\alpha,\beta-1)}\right)' \stackrel{\frac{d}{d\mu}(4)}{=} 2\mu\left(\phi_n^{(\alpha,\beta)}\right)' + \frac{n+\alpha}{n+\alpha+\beta} 2\mu\left(\phi_{n-1}^{(\alpha,\beta)}\right)'. \quad (16)$$

The key role further plays the following combination of (10) and (11), which is valid for an arbitrary real A ,

$$\frac{n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta)} + A\phi_{n-1}^{(\alpha,\beta)} = \frac{(1+A)n+\alpha+\beta}{n+\alpha}\phi_n^{(\alpha,\beta-1)} + 2\mu\frac{1-A}{n+\alpha}\left(\phi_n^{(\alpha,\beta-1)}\right)'. \quad (17)$$

The next identity stands apart and can be checked explicitly with the help of (2)

$$\phi_n^{(\alpha,\beta)}(\mu) - \phi_n^{(\alpha,\beta)}(0) = \frac{1}{4}(n+\alpha+\beta+1)_2 \cdot \mu\phi_{n-2}^{(\alpha+2,\beta+2)}(\mu).$$

It reflects the standard formula for the derivative of the Jacobi polynomial (e.g. [6, p. 737]):

$$\left(P_n^{(\alpha,\beta)}(x)\right)^{(m)} = 2^{-m}(n+\alpha+\beta+1)_m P_{n-m}^{(\alpha+m,\beta+m)}(x). \quad (18)$$

Remark. Note that the equalities (4)–(18) are of formal nature, and therefore their validity requires no orthogonality from the Jacobi polynomials. That is, these equalities holds true if all coefficients are defined, not only for $\alpha, \beta > -1$.

Remark. A polynomial with only real zeros interlaces its derivative by Rolle's theorem.¹ Consequently, they both interlace any real combination of them.² So if in one of the formulae (10)–(12) the first term on the right-hand side has only real zeros, then all three involved polynomials are pairwise interlacing. For example, if $\phi_n^{(\alpha,\beta)}$ has only real zeros, then $\phi_{n-1}^{(\alpha,\beta+1)}$, $\phi_n^{(\alpha,\beta)}$ and $2\mu\left(\phi_n^{(\alpha,\beta)}\right)'$ are pairwise interlacing which is provided by (10).

Remark. The identities (6) and (7) show that the interlacing property of $\phi_n^{(\alpha,\beta)}(\mu)$ and $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ can also be expressed as the interlacing property of $\phi_n^{(\alpha,\beta-1)}(\mu)$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$. Analogously, from the relations (13) and (14) it can be seen that this is also equivalent to the interlacing property of $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ and $\phi_n^{(\alpha-1,\beta)}(\mu)$.

Lemma 1. *If $-1 < \alpha < 1$, $-1 < \beta$ or if $1 \leq \alpha < 2$, $0 < \beta$, then the polynomials $\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$, $\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)'$ and $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ are pairwise interlacing.*

Proof. By Theorem CCW the polynomials $\phi_n^{(\alpha,\beta)}(\mu)$, $\phi_{n-1}^{(\alpha,\beta)}(\mu)$ have only (simple) negative zeros. Then the relation (10) shows that the polynomial $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ with negative zeros interlaces (strictly) both $\phi_n^{(\alpha,\beta)}(\mu)$ and $\mu\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$. Moreover, the sign of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ at the origin and at the rightmost zero of $\phi_n^{(\alpha,\beta)}(\mu)$ is the same, and therefore $\left(\phi_n^{(\alpha,\beta)}(\mu), \mu\phi_{n-1}^{(\alpha,\beta+1)}(\mu)\right)$ is a real coprime pair. A similar consideration of the relation (11) gives that the pair $\left(\mu\phi_{n-1}^{(\alpha,\beta)}(\mu), \phi_{n-1}^{(\alpha,\beta+1)}(\mu)\right)$ is also real and coprime.

In particular, each interval between two consequent zeros of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ contains exactly one zero of $\phi_n^{(\alpha,\beta)}(\mu)$

¹Non-strictly whenever the polynomial has a multiple zero.

²See the definition of a real pair on the page 2.

as well as of $\phi_{n-1}^{(\alpha,\beta)}(\mu)$. Taking the signs of the last two polynomials at the ends of these intervals into account shows that the difference in the left-hand side of (16), and hence $\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$, changes its sign between consecutive zeros of $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Since $\deg\left(\phi_n^{(\alpha,\beta-1)}\right)' \leq \deg\phi_{n-1}^{(\alpha,\beta+1)}$, the polynomial $\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$ necessarily interlaces $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$. Then the right-hand side of (16) shows that

$$\left(\phi_n^{(\alpha,\beta)}(\mu)\right)' + \frac{n+\alpha}{n+\alpha+\beta}\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)' \quad (19)$$

and $\phi_{n-1}^{(\alpha,\beta+1)}(\mu)$ have interlacing zeros. At the same time, by differentiating the equality (16) we obtain that

$$n\left(\phi_n^{(\alpha,\beta)}(\mu)\right)' - (n+\alpha)\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)' \quad (20)$$

is proportional to $\left(\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'\right)'$ and, hence, interlaces $\mu\left(\phi_n^{(\alpha,\beta-1)}(\mu)\right)'$. Put in other words, the polynomials (19) and (20) are interlacing. With appropriate factors, their sum gives $\left(\phi_n^{(\alpha,\beta)}(\mu)\right)'$ and their difference gives $\left(\phi_{n-1}^{(\alpha,\beta)}(\mu)\right)'$. This yields the lemma. \square

3 Results on Conjecture A

Theorem HB (Hermite-Biehler, for real polynomials). *A real polynomial $f(z) := p(z^2) + zq(z^2)$ is stable³ if and only if $p(0) \cdot q(0) > 0$ and all zeros of $p(z)$ and $zq(z)$ are nonpositive and strictly interlacing.*

This well-known fact plays a crucial role in [4] for proving Theorem CCW. Here the statement of the Hermite-Biehler theorem is expressed closely to the one given in [5, p. 228]. The replacement of $q(z)$ with $zq(z)$ provides the desired order of zeros. That is, the zero of $p(z)$ and $q(z)$ closest to the origin belongs to the former polynomial. The more general statement can be found in e.g. [2, p. 21]. The present study also uses Theorem HB as a main tool.

Some bounds on the parameters α and β are necessary even for Theorem CCW (i.e. are satisfied if all zeros of $\phi_n^{(\alpha,\beta)}(\mu)$ are negative for every $n \geq 2$). The restriction $\alpha > -1$ corresponds to positivity of the coefficients (and to negativity of all zeros). The parameter α is bounded from above by 3.37228.... Indeed, if the polynomial $\phi_n^{(\alpha,\beta)}(\mu) =: \sum_{k=0}^m b_k \mu^k$, $m = \lfloor n/2 \rfloor$, has only negative⁴ zeros, then by Rolle's theorem, the zeros of $p(\mu) := \mu^m \phi_n^{(\alpha,\beta)}(\mu^{-1})$ and $p'(\mu)$ are interlacing. So, Theorem HB then implies that the polynomial $p(\mu^2) + \mu p'(\mu^2)$ is stable. Therefore, when $b_0 > 0$ the Hurwitz criterion (see e.g. [2, Chapter I]) gives us the easy-to-check conditions $b_1 > 0$ and

$$\begin{vmatrix} mb_0 & (m-1)b_1 & (m-2)b_2 \\ b_0 & b_1 & b_2 \\ 0 & mb_0 & (m-1)b_1 \end{vmatrix} \geq 0, \quad \text{that is} \quad \frac{b_1^2}{b_0 b_2} \geq \frac{2m}{m-1} > 2, \quad (21)$$

which are necessarily true when the polynomial $p(\mu)$ has only real zeros. In our case we have

$$\frac{b_1^2}{b_0 b_2} = \frac{n(n-1)(\alpha+3)_2(n+\alpha+\beta+1)_2}{(n-2)(n-3)(\alpha+1)_2(n+\alpha+\beta+3)_2} \xrightarrow{n \rightarrow \infty} \frac{(\alpha+3)(\alpha+4)}{(\alpha+1)(\alpha+2)},$$

so the condition (21) fails to be true (along with the assertion of Theorem CCW) for every n big enough and every β when $\alpha > \frac{1+\sqrt{33}}{2} = 3.37228\dots$ or $\alpha < \frac{1-\sqrt{33}}{2} = -2.37228\dots$. Computer experiments show that the polynomials $\phi_n^{(\alpha,\beta)}$ with *positive coefficients* can have zeros outside the real axis for a (large enough) negative β . So, some lower constraint on the parameter β is also required.

³That is, Hurwitz stable: $f(z) = 0 \implies \operatorname{Re} z < 0$.

⁴Here we suppose that $\phi_n^{(\alpha,\beta)}(\mu)$ has only simple zeros; the case of multiple zeros follows by continuity.

Definition. Denote the i th zero of a polynomial p with respect to the distance from the origin by $\mathbf{zr}_i(p)$. Put $\mathbf{zr}_i(p)$ equal to $-\infty$ if $\deg p < i$ and to zero if $i = 0$ (it is convenient since all coefficients of the polynomials we deal with are nonnegative).

Lemma 2. Let $\alpha > -1$ and $n + \alpha + \beta > 0$, $n = 4, 5, \dots$. The zeros of the polynomials $\phi_n^{(\alpha, \beta)}$ and $\phi_{n-1}^{(\alpha, \beta)}$ are negative and interlace non-strictly (strictly) if and only if the polynomial $\phi_n^{(\alpha, \beta-1)}$ has only real zeros (only simple real zeros, respectively). Moreover, if $\phi_n^{(\alpha, \beta-1)}$ has only real zeros, then $\mathbf{zr}_1(\phi_{n-1}^{(\alpha, \beta)}) \leq \mathbf{zr}_1(\phi_n^{(\alpha, \beta)})$.

Proof. The relation (17) with $A = 1$ and $A = -(n + \alpha + \beta)/n$ implies that each common zero of the polynomials $\phi_{n-1}^{(\alpha, \beta)}$ and $\phi_n^{(\alpha, \beta)}$ is a multiple zero of $\phi_n^{(\alpha, \beta-1)}$. The converse result is given by (10) and (11).

Suppose that $\phi_n^{(\alpha, \beta-1)}$ has only real zeros. The coefficients of this polynomial are positive under the assumptions of the lemma, and hence all of its zeros are negative. By Rolle's theorem, the pair $(\phi_n^{(\alpha, \beta-1)}, \mu(\phi_n^{(\alpha, \beta-1)}))$ is real. Therefore, the polynomials $\phi_{n-1}^{(\alpha, \beta)}$ and $\phi_n^{(\alpha, \beta)}$ have only real zeros by the formulae (10) and (11), respectively. The zeros are negative automatically since the coefficients of polynomials are positive. Moreover, we have that the first zero of $\phi_n^{(\alpha, \beta)}$ is closer to the origin than that of $\phi_{n-1}^{(\alpha, \beta)}$. The relation (17) holds for all real A , which yields that the polynomials $\phi_{n-1}^{(\alpha, \beta)}$ and $\phi_n^{(\alpha, \beta)}$ form a real pair and thus have (non-strictly) interlacing zeros.

Let $\phi_{n-1}^{(\alpha, \beta)}$ and $\phi_n^{(\alpha, \beta)}$ have negative interlacing zeros. Then any of their real combinations only has real zeros. This is true for $\phi_n^{(\alpha, \beta-1)}$ according to the identity (4). \square

Corollary 3. For $-1 < \alpha < 1$, $\beta > 0$ or $1 \leq \alpha < 2$, $\beta > 1$ the zeros of the polynomials $\phi_n^{(\alpha, \beta)}$ and $\phi_{n-1}^{(\alpha, \beta)}$, $n = 4, 5, \dots$, interlace. (Furthermore, the polynomials $\phi_n^{(\alpha, \beta)}$ and $\mu\phi_{n-1}^{(\alpha, \beta)}$ interlace.)

Proof. This corollary is provided by Theorem CCW (see also the remark on it) and Lemma 2. \square

Theorem CW (Theorem 3.10, Charalambides et al. [3]). Conjecture A holds true for $-1 < \alpha, \beta < 0$ and for $0 < \alpha, \beta < 1$.

This theorem relies on Theorem 3.8 and Theorem 3.9 of the same work and on Theorem CCW. In fact, the original proof (which we extend in the next section to treat Conjecture B) does not need any upper bound on β . It becomes more evident on recalling that the region of positive β is covered by Corollary 3 (i.e. is a straightforward consequence of Lemma 2 and Theorem CCW).

Theorem CW*. Conjecture A holds true for $-1 < \alpha < 0$, $-1 < \beta$ or $0 \leq \alpha < 1$, $0 < \beta$ or $1 \leq \alpha < 2$, $1 < \beta$.

Proof. Corollary 3 suits the case of positive α . For the region with negative α it is enough to prove only Theorem 4 (which is stated below) and Theorem 3.9 is not needed. Indeed, according to Theorem 4 and Theorem HB we have that all zeros of $\phi_n^{(\alpha, \beta-1)}$ are simple and real for all n , so Lemma 2 is applicable. \square

Theorem 4 (Equivalent to Theorem 3.8, Charalambides et al. [3]). If $-1 < \alpha < 0$, $-1 < \beta$ and $n = 4, 5, \dots$, then the zeros of the polynomial $\Phi_n(1; \mu)$, where

$$\Phi_n(x; \mu) := \sum_{k=0}^n \frac{d^k}{dx^k} P_n^{(\alpha, \beta-1)}(x) \mu^k,$$

lie in the open left half of the complex plane.

Remark. It is worth noting that this theorem cannot be extended to the full range $-1 < \beta < 0 < \alpha < 1$. According to computer experiments, Conjecture A seems to hold in this range, while Theorem 4 fails, e.g., for $n = 12$ when $\beta = -0.8$ and $\alpha \gtrapprox 0.97842$, or when $\beta = -0.9$ and $\alpha \gtrapprox 0.97140$. The reason is that proving Conjecture A only requires negative simple zeros of the polynomial $\phi_n^{(\alpha, \beta-1)}(\mu)$, Theorem 4 nevertheless asserts additional properties of $\phi_{n-1}^{(\alpha+1, \beta)}(\mu)$ as given by the Hermite-Biehler theorem.

Proof. This proof is akin to [3, Theorem 3.8] but uses other relations for the Jacobi polynomials. The polynomial $\Phi_n(x; \mu)$ satisfies the differential equation (here we consider μ as a parameter)

$$\Phi_n(x; \mu) = P_n^{(\alpha, \beta-1)}(x) + \mu \frac{d\Phi_n(x; \mu)}{dx}.$$

Let $\Phi_n := \Phi_n(x; \mu)$ for brevity and let $\overline{\frac{d\Phi_n}{dx}}$ denote a complex conjugate of $\frac{d\Phi_n}{dx}$. Multiplying by $\overline{\frac{d\Phi_n}{dx}} w_{\alpha, \beta+1}$, where $w_{\alpha, \beta+1} := w_{\alpha, \beta+1}(x) = (1-x)^\alpha(1+x)^{\beta+1}$, and integration over the interval $(-1, 1)$ gives us

$$\int_{-1}^1 \Phi_n \overline{\frac{d\Phi_n}{dx}} w_{\alpha, \beta+1} dx = \int_{-1}^1 P_n^{(\alpha, \beta-1)} \overline{\frac{d\Phi_n}{dx}} w_{\alpha, \beta+1} dx + \mu \int_{-1}^1 \left| \frac{d\Phi_n}{dx} \right|^2 w_{\alpha, \beta+1} dx.$$

Select μ so that $\Phi_n(1; \mu) = 0$. To estimate the real part of μ we add the last equation to its complex conjugate and obtain

$$\int_{-1}^1 \frac{d(|\Phi_n|^2)}{dx} w_{\alpha, \beta+1} dx = \int_{-1}^1 P_n^{(\alpha, \beta-1)} \cdot 2 \operatorname{Re} \frac{d\Phi_n}{dx} \cdot w_{\alpha, \beta+1} dx + 2 \operatorname{Re} \mu \int_{-1}^1 \left| \frac{d\Phi_n}{dx} \right|^2 w_{\alpha, \beta+1} dx. \quad (22)$$

Since $w_{\alpha, \beta+1}$ increases on $(-1, 1)$ and $\lim_{x \rightarrow -1+} \Phi_n w_{\alpha, \beta+1} = \lim_{x \rightarrow -1-} \Phi_n w_{\alpha, \beta+1} = 0$, the left-hand side satisfies

$$\int_{-1}^1 \frac{d(|\Phi_n|^2)}{dx} w_{\alpha, \beta+1} dx = - \int_{-1}^1 |\Phi_n|^2 w'_{\alpha, \beta+1} dx < 0.$$

Applying the relation (8) to the polynomial $P_n^{(\alpha, \beta-1)}$ three times gives us

$$\begin{aligned} P_n^{(\alpha, \beta-1)} &= \frac{n+\alpha+\beta}{2n+\alpha+\beta} P_n^{(\alpha, \beta)} + \frac{n+\alpha}{2n+\alpha+\beta} P_{n-1}^{(\alpha, \beta)} = \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2} P_n^{(\alpha, \beta+1)} + \frac{(n+\alpha+\beta)(n+\alpha)}{(2n+\alpha+\beta)_2} P_{n-1}^{(\alpha, \beta+1)} \\ &\quad + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)_2} P_{n-1}^{(\alpha, \beta+1)} + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2} P_{n-2}^{(\alpha, \beta+1)}, \end{aligned}$$

that is,

$$P_n^{(\alpha, \beta-1)} = \frac{(n+\alpha+\beta)_2}{(2n+\alpha+\beta)_2} P_n^{(\alpha, \beta+1)} + \frac{2(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha, \beta+1)} + \frac{(n+\alpha-1)_2}{(2n+\alpha+\beta-1)_2} P_{n-2}^{(\alpha, \beta+1)}.$$

By the definition of Φ_n and the formula (18),

$$\operatorname{Re} \frac{d\Phi_n}{dx} = 2^{-1}(n+\alpha+\beta) P_{n-1}^{(\alpha+1, \beta)} + \operatorname{Re} \mu \cdot 2^{-2}(n+\alpha+\beta)_2 P_{n-2}^{(\alpha+2, \beta+1)} + \{\text{polynomials of degree } < n-2\}.$$

The difference (8) – (9) induces the identity $P_{n-1}^{(\alpha+1, \beta)} = P_{n-1}^{(\alpha, \beta+1)} + P_{n-2}^{(\alpha+1, \beta+1)}$, so we finally have

$$\int_{-1}^1 P_n^{(\alpha, \beta-1)} \cdot 2 \operatorname{Re} \frac{d\Phi_n}{dx} \cdot w_{\alpha, \beta+1} dx = \eta + \zeta \operatorname{Re} \mu, \quad \text{where } \eta, \zeta > 0.$$

Now the terms of the relation (22) are estimated, and it yields $0 > \operatorname{Re} \mu$. □

4 Results on Conjecture B

We just have shown that the result on the Conjecture A in [3] can be obtained in a shorter way if we consider polynomials with shifted parameter values (we used $\phi_n^{(\alpha, \beta-1)}$) instead of real combinations of the polynomials $\phi_n^{(\alpha, \beta)}$ and $\phi_{n-1}^{(\alpha, \beta)}$. At the same time, to verify the Conjecture B we can combine both these ideas.

For any fixed $n > 3$ consider the intermediary polynomial

$$f(x; \mu) := \sum_{k=0}^n \mu^k \left(A \frac{d^k}{dx^k} P_n^{(\alpha, \beta)}(x) + \mu \frac{d^k}{dx^k} P_{n-1}^{(\alpha, \beta)}(x) \right).$$

Lemma 5. *The polynomial $f(1; \mu)$ is Hurwitz-stable provided that $-1 < \alpha < 1$ and $\beta, A > 0$.*

Proof. From the definition of $f(x, \mu)$ the differential equation

$$\mu \frac{d}{dx} f(x; \mu) + AP_n^{(\alpha, \beta)}(x) + \mu P_{n-1}^{(\alpha, \beta)}(x) = f(x; \mu)$$

follows. Multiplication by $\overline{f(x; \mu)} w_{\alpha-1, \beta}(x)$ gives us

$$\overline{f} \frac{df}{dx} w_{\alpha-1, \beta} + AP_n^{(\alpha, \beta)}(x) \frac{\overline{f}}{\mu} w_{\alpha-1, \beta} + P_{n-1}^{(\alpha, \beta)}(x) \overline{f} w_{\alpha-1, \beta} = \frac{1}{\mu} |f|^2 w_{\alpha-1, \beta},$$

where we put $f := f(x; \mu)$ and $w_{\alpha-1, \beta} := w_{\alpha-1, \beta}(x) = (1-x)^{\alpha-1}(1+x)^\beta$ for brevity. Adding to this equality its complex conjugate and integrating yields

$$\begin{aligned} \int_{-1}^1 \frac{d(|f|^2)}{dx} w_{\alpha-1, \beta} dx + A \int_{-1}^1 P_n^{(\alpha, \beta)}(x) \left(\frac{\overline{f}}{\mu} + \frac{f}{\overline{\mu}} \right) w_{\alpha-1, \beta} dx + \int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(x) (\overline{f} + f) w_{\alpha-1, \beta} dx \\ = \left(\frac{1}{\mu} + \frac{1}{\overline{\mu}} \right) \int_{-1}^1 |f|^2 w_{\alpha-1, \beta} dx. \end{aligned} \quad (23)$$

Take μ so that $f(1; \mu) = 0$. Then the polynomial $f(x; \mu)$ can be represented as

$$f(x; \mu) = (1-x) \sum_{k=0}^{n-1} c_k P_k^{(\alpha, \beta)}(x)$$

with some complex constants c_k depending on μ (generally speaking). Observe that $c_{n-1} < 0$: denoting the leading coefficient in x by \mathbf{lc} , we obtain

$$c_{n-1} = \frac{\mathbf{lc}(f(x; \mu))}{\mathbf{lc}(-x P_{n-1}^{(\alpha, \beta)}(x))} = -\frac{A \cdot \mathbf{lc}(P_n^{(\alpha, \beta)}(x))}{\mathbf{lc}(P_{n-1}^{(\alpha, \beta)}(x))} \stackrel{(1)}{=} -A \frac{(n+\alpha+\beta+1)_n \cdot (n-1)! 2^{n-1}}{n! 2^n \cdot (n+\alpha+\beta)_{n-1}} = -A \frac{(2n+\alpha+\beta-1)_2}{2n(n+\alpha+\beta)} < 0.$$

Then we have

$$\begin{aligned} \int_{-1}^1 P_n^{(\alpha, \beta)}(x) f w_{\alpha-1, \beta} dx &= \int_{-1}^1 P_n^{(\alpha, \beta)}(x) \sum_{k=0}^{n-1} c_k P_k^{(\alpha, \beta)}(x) w_{\alpha, \beta} dx = 0, \\ \int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(x) f w_{\alpha-1, \beta} dx &= c_{n-1} \int_{-1}^1 (P_{n-1}^{(\alpha, \beta)}(x))^2 w_{\alpha, \beta} dx < 0 \end{aligned} \quad (24)$$

as a consequence of orthogonality of the Jacobi polynomials. Note that if $-1 < \alpha < 1$ and $\beta > 0$, then

$$w'_{\alpha-1, \beta} = (\beta(1-x) - (\alpha-1)(1+x)) \cdot w_{\alpha-2, \beta-1} > 0 \quad \text{and} \quad \lim_{x \rightarrow -1+} |f|^2 w_{\alpha-1, \beta} = \lim_{x \rightarrow 1-} |f|^2 w_{\alpha-1, \beta} = 0.$$

Therefore, integrating by parts yields

$$\int_{-1}^1 \frac{d(|f|^2)}{dx} w_{\alpha-1, \beta} dx = - \int_{-1}^1 (|f|^2) w'_{\alpha-1, \beta} dx < 0. \quad (25)$$

By the formulae (24)–(25), the left-hand side of the equation (23) is negative, thus necessarily $\operatorname{Re} \mu < 0$. \square

Consider also another intermediary polynomial for a fixed $n > 3$, namely

$$g(x; \mu) := \sum_{k=0}^n \mu^k \left(\mu \frac{d^k}{dx^k} P_n^{(\alpha, \beta)}(x) + A \frac{d^k}{dx^k} P_{n-1}^{(\alpha, \beta)}(x) \right).$$

Lemma 6. *The polynomial $g(1; \mu)$ is Hurwitz-stable provided that $-1 < \alpha < 0$ and $\beta, A > 0$.*

Proof. This proof is analogous to the proofs of Theorem 4 and Lemma 5. From the definition of $g(x, \mu)$ we have

$$\mu \frac{dg(x; \mu)}{dx} + \mu P_n^{(\alpha, \beta)}(x) + AP_{n-1}^{(\alpha, \beta)}(x) = g(x; \mu)$$

which gives us (we put $g := g(x; \mu)$ and $g' := \frac{dg(x; \mu)}{dx}$ for brevity's sake)

$$\mu g' \overline{g'} w_{\alpha, \beta} + \mu P_n^{(\alpha, \beta)}(x) \overline{g'} w_{\alpha, \beta} + AP_{n-1}^{(\alpha, \beta)}(x) \overline{g'} w_{\alpha, \beta} = g \overline{g'} w_{\alpha, \beta}$$

after multiplication by $\overline{g'} w_{\alpha, \beta}$. Adding to the equation its complex conjugate and integrating yields

$$\begin{aligned} (\mu + \overline{\mu}) \int_{-1}^1 |g'|^2 w_{\alpha, \beta} dx + \int_{-1}^1 P_n^{(\alpha, \beta)}(x) (\mu \overline{g'} + \overline{\mu} g') w_{\alpha, \beta} dx + A \int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(x) (\overline{g'} + g') w_{\alpha, \beta} dx \\ = \int_{-1}^1 (g \overline{g'} + \overline{g} g') w_{\alpha, \beta} dx. \end{aligned} \quad (26)$$

Observe that g is a polynomial of degree n in x and its leading coefficient is given by $P_n^{(\alpha, \beta)}(x) \cdot \mu$. Consequently, substituting the explicit expression for $\mathbf{lc}(P_n^{(\alpha, \beta)}(x))$ from the formula (1) gives

$$\mathbf{lc}(g') = n \mathbf{lc}(P_n^{(\alpha, \beta)}(x)) \mu = \frac{(2n + \alpha + \beta - 1)_2}{2(n + \alpha + \beta)} \cdot \frac{(n + \alpha + \beta)_{n-1}}{(n-1)! 2^{n-1}} \mu = \frac{(2n + \alpha + \beta - 1)_2}{2(n + \alpha + \beta)} \mathbf{lc}(P_{n-1}^{(\alpha, \beta)}(x)) \mu.$$

This allows us to calculate the third summand on the left-hand side of (26):

$$\begin{aligned} \int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(x) (\overline{g'} + g') w_{\alpha, \beta} dx &= \int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(x) \frac{(2n + \alpha + \beta - 1)_2}{2(n + \alpha + \beta)} \mathbf{lc}(P_{n-1}^{(\alpha, \beta)}(x)) (\overline{\mu} + \mu) x^{n-1} w_{\alpha, \beta} dx \\ &= (\overline{\mu} + \mu) \frac{(2n + \alpha + \beta - 1)_2}{2(n + \alpha + \beta)} \int_{-1}^1 (P_{n-1}^{(\alpha, \beta)}(x))^2 w_{\alpha, \beta} dx. \end{aligned} \quad (27)$$

Additionally, we have

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(x) g' w_{\alpha, \beta} dx = 0. \quad (28)$$

Take μ so that $g(1; \mu) = 0$. Then

$$w'_{\alpha, \beta} = (\beta(1-x) - \alpha(1+x)) \cdot w_{\alpha-1, \beta-1} > 0 \quad \text{and} \quad \lim_{x \rightarrow -1+} |g|^2 w_{\alpha, \beta} = \lim_{x \rightarrow -1-} |g|^2 w_{\alpha, \beta} = 0$$

since $-1 < \alpha < 0$ and $\beta > 0$. Integrating by parts we obtain

$$\int_{-1}^1 (g \overline{g'} + \overline{g} g') w_{\alpha, \beta} dx = - \int_{-1}^1 |g|^2 w'_{\alpha, \beta} dx < 0. \quad (29)$$

Now let us bring together the relations (26)–(29):

$$(\mu + \overline{\mu}) \int_{-1}^1 \left(|g'|^2 + A \frac{(2n + \alpha + \beta - 1)_2}{2(n + \alpha + \beta)} (P_{n-1}^{(\alpha, \beta)}(x))^2 \right) w_{\alpha, \beta} dx = - \int_{-1}^1 |g|^2 w'_{\alpha, \beta} dx, \quad \text{hence} \quad 2 \operatorname{Re} \mu = \mu + \overline{\mu} < 0.$$

That is, any zero μ of the polynomial $g(1; \mu)$ resides in the left half of the complex plane. \square

Corollary 7. For any positive A , the zeros of the polynomials

$$2A\phi_n^{(\alpha, \beta)}(\mu) + (n + \alpha + \beta) \mu \phi_{n-2}^{(\alpha+1, \beta+1)}(\mu) \quad \text{and} \quad A(n + \alpha + \beta + 1) \phi_{n-1}^{(\alpha+1, \beta+1)}(\mu) + 2\phi_{n-1}^{(\alpha, \beta)}(\mu) \quad (30)$$

are interlacing provided that $-1 < \alpha < 1$ and $\beta > 0$. If in addition $-1 < \alpha < 0$, the zeros of the polynomials

$$(n + \alpha + \beta + 1) \mu \phi_{n-1}^{(\alpha+1, \beta+1)}(\mu) + 2A\phi_{n-1}^{(\alpha, \beta)}(\mu) \quad \text{and} \quad 2\phi_n^{(\alpha, \beta)}(\mu) + A(n + \alpha + \beta) \phi_{n-2}^{(\alpha+1, \beta+1)}(\mu) \quad (31)$$

are also interlacing.

Proof. To get the assertion we apply the relation (18) to the even and odd parts of the polynomials $f(x; \mu)$ and $g(x; \mu)$. The even part of $f(x; \mu)$ is

$$\begin{aligned} \frac{f(x; \mu) + f(x; -\mu)}{2} &= A \sum_{k=0}^{[n/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha, \beta)}(x) + \mu^2 \sum_{k=0}^{[n/2]} \mu^{2k} \frac{d^{2k+1}}{dx^{2k+1}} P_{n-1}^{(\alpha, \beta)}(x) \\ &= A \sum_{k=0}^{[n/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha, \beta)}(x) + \frac{n + \alpha + \beta}{2} \mu^2 \sum_{k=0}^{[n/2]-1} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_{n-2}^{(\alpha+1, \beta+1)}(x). \end{aligned}$$

Analogously, for the odd part we have

$$\begin{aligned} \frac{f(x; \mu) - f(x; -\mu)}{2} &= A \sum_{k=0}^{[(n-1)/2]} \mu^{2k+1} \frac{d^{2k+1}}{dx^{2k+1}} P_n^{(\alpha, \beta)}(x) + \sum_{k=0}^{[(n-1)/2]} \mu^{2k+1} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha, \beta)}(x) \\ &= A \frac{n + \alpha + \beta + 1}{2} \sum_{k=0}^{[(n-1)/2]} \mu^{2k+1} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha+1, \beta+1)}(x) + \sum_{k=0}^{[(n-1)/2]} \mu^{2k+1} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha, \beta)}(x). \end{aligned}$$

The same manipulations with $g(x; \mu)$ give

$$\begin{aligned} \frac{g(x; \mu) + g(x; -\mu)}{2} &= \sum_{k=0}^{[(n-1)/2]} \mu^{2k+2} \frac{d^{2k+1}}{dx^{2k+1}} P_n^{(\alpha, \beta)}(x) + A \sum_{k=0}^{[(n-1)/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha, \beta)}(x) \\ &= \frac{n + \alpha + \beta + 1}{2} \mu^2 \sum_{k=0}^{[(n-1)/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha+1, \beta+1)}(x) + A \sum_{k=0}^{[(n-1)/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_{n-1}^{(\alpha, \beta)}(x) \quad \text{and} \\ \frac{g(x; \mu) - g(x; -\mu)}{2} &= \sum_{k=0}^{[n/2]} \mu^{2k+1} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha, \beta)}(x) + A \sum_{k=0}^{[(n-1)/2]} \mu^{2k+1} \frac{d^{2k+1}}{dx^{2k+1}} P_{n-1}^{(\alpha, \beta)}(x) \\ &= \mu \left(\sum_{k=0}^{[n/2]} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_n^{(\alpha, \beta)}(x) + A \frac{n + \alpha + \beta}{2} \sum_{k=0}^{[n/2]-1} \mu^{2k} \frac{d^{2k}}{dx^{2k}} P_{n-2}^{(\alpha+1, \beta+1)}(x) \right). \end{aligned}$$

The polynomials $f(1; \mu)$ and $g(1; \mu)$ are stable by Lemma 5 and Lemma 6, respectively. Thus, the pairs of polynomials mentioned in (30) and (31) have interlacing zeros by Theorem HB. \square

Lemma 8 (see e.g. [8, Lemma 3.4] or [3, Lemma 3.5]). *Let real polynomials $p(x)$ and $q(x)$ such that $p(0), q(0) > 0$ have only negative zeros. Then $(p(x), xq(x))$ is a real pair if and only if the combinations*

$$r_1(x) := r_1(x; A, B) := Ap(x) + Bxq(x) \quad \text{and} \quad r_2(x) := r_2(x; C, D) := Cp(x) + Dq(x) \quad (32)$$

are nonzero outside the real line for all $A, B, C, D > 0$.

Recall that $p(x)$ and $xq(x)$ is a real pair whenever they interlace (non-strictly if $p(x)$ and $xq(x)$ have a common zero). For p and q as in this lemma we thus have $\deg p \leq \deg q \leq 1 + \deg p$ automatically.

Proof. Without loss of generality, assume in the proof that the polynomials p and q have no common zeros: if not, the zeros are real and we can factor them out of r_1 and r_2 . The presence of common zeros prevents p and q from being strictly interlacing.

Let $(p(x), xq(x))$ be a real pair. The polynomials p and xq interlace exactly when between each pair of consecutive zeros $\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(p)$ the polynomial q has exactly one change of sign, $i = 2, \dots, \deg p$. That is, the interlacing property of these polynomials is equivalent to

$$R(z) := \frac{q(z)}{p(z)} = \gamma + \sum_{i=1}^n \frac{A_i}{z - \mathbf{zr}_i(p)} \quad \text{implying} \quad zR(z) = \frac{zq(z)}{p(z)} = \gamma z + \sum_{i=1}^n A_i + \sum_{i=1}^n \frac{\mathbf{zr}_i(p) A_i}{z - \mathbf{zr}_i(p)},$$

where $\gamma \geq 0$ and the residues $A_i = \frac{q(\mathbf{zr}_i(p))}{p'(\mathbf{zr}_i(p))}$ are positive for all i . The straightforward check shows that

$$\text{sign Im } R(z) = -\text{sign Im } (zR(z)) = -\text{sign Im } z$$

for any $z \in \mathbb{C}$ such that $p(z) \neq 0$. Consequently, the combinations (32) have zeros only on the real line.

Conversely, let for any fixed $A, B, C, D > 0$ the polynomials $r_1(x; A, B)$ and $r_2(x; C, D)$ have only real zeros. The zeros of r_1, r_2 are all negative since all their coefficients are positive. Moreover, $p(x)$ and $q(x)$ are coprime, and therefore

$$\begin{aligned} r_1(x_*; A, B) = 0 &\implies \text{sign } p(x_*) = \text{sign } q(x_*) = \text{sign } r_2(x_*; C, D) \neq 0 \quad \text{and} \quad r_1(x_*; \tilde{A}, \tilde{B}) \neq 0, \\ r_2(x_{\#}; C, D) = 0 &\implies \text{sign } p(x_{\#}) = -\text{sign } q(x_{\#}) = \text{sign } r_1(x_{\#}; A, B) \neq 0 \quad \text{and} \quad r_2(x_{\#}; \tilde{C}, \tilde{D}) \neq 0, \end{aligned}$$

where $B\tilde{A} \neq A\tilde{B}$ and $D\tilde{C} \neq C\tilde{D}$.

The roots of a polynomial depend continuously on its coefficients. Therefore, when the ratio B/A comes close to zero (or to infinity), the roots of r_1 tend to the roots of $p(x)$ (or to the roots of $xq(x)$, respectively). For $i = 1, \dots, \deg r_1$ and $A, B > 0$ the zero $\mathbf{zr}_i(r_1)$ can never coincide with a root of $p(x)$ or $xq(x)$ thus remaining in the interval

$$I_i := \bigcup_{A/B > 0} \mathbf{zr}_i(r_1) = (\min\{\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(q)\}, \max\{\mathbf{zr}_i(p), \mathbf{zr}_{i-1}(q)\}).$$

When $k := \deg p - \deg q - 1 \neq 0$ there exist $|k|$ surplus roots of $r_1(x)$ which disappear as $r_1(x)$ becomes proportional to $p(x)$ or $xq(x)$. Being negative, these roots must tend to $-\infty$. Since they can never meet a root of $p(x)$ or $q(x)$, they run the whole ray $(-\infty, \mathbf{zr}_{\deg p}(p))$ if $k > 0$ and $(-\infty, \mathbf{zr}_{\deg q}(q))$ if $k < 0$. This implies $|k| \leq 1$, because $r_1(x; A, B)$ and $r_1(x; \tilde{A}, \tilde{B})$ have no common zeros unless $B/A = \tilde{B}/\tilde{A}$.

For the polynomial r_2 we analogously obtain

$$J_i := \bigcup_{C/D > 0} \mathbf{zr}_i(r_2) = (\min\{\mathbf{zr}_i(q), \mathbf{zr}_i(p)\}, \max\{\mathbf{zr}_i(q), \mathbf{zr}_i(p)\})$$

and $|\deg p - \deg q| \leq 1$. The last inequality together with $|k| \leq 1$ gives $\deg q \leq \deg p \leq \deg q + 1$. For each $z < 0$ there exists $i = 1, \dots, \deg q + 1$ such that $z \in [\mathbf{zr}_i(q), \mathbf{zr}_{i-1}(q)] \subset \bar{J}_i \cup I_i$. Since J_i and I_i are disjoint, the only option is $\mathbf{zr}_i(q) \leq \mathbf{zr}_i(p) \leq \mathbf{zr}_{i-1}(q)$. This implies that the polynomial $p(x)$ interlace $xq(x)$ since $\deg p \leq \deg q + 1$. \square

The next corollary complements the interlacing property of the polynomials $\phi_n^{(\alpha, \beta)}(\mu)$ and $\phi_{n-1}^{(\alpha+1, \beta+1)}(\mu)$ (see the remark to Theorem CCW).

Corollary 9. *If $-1 < \alpha < 0$ and $\beta > 0$ the pairs $(\phi_n^{(\alpha, \beta)}(\mu), \mu\phi_{n-2}^{(\alpha+1, \beta+1)}(\mu))$ and $(\phi_n^{(\alpha, \beta)}(\mu), \mu\phi_n^{(\alpha+1, \beta+1)}(\mu))$ possess the (strict) interlacing property.*

Proof. By Theorem CCW, all involved polynomials have only real nonpositive zeros. Corollary 7 adds that the polynomials in (30)–(31) have (strictly) interlacing zeros. Therefore, Lemma 8 assures the asserted fact. \square

Theorem 10. *If $-1 < \alpha < 0 < \beta$ and $n = 5, 6, \dots$, then the polynomial $\phi_n^{(\alpha+1, \beta+1)}(\mu)$ interlaces $\phi_{n-2}^{(\alpha+1, \beta+1)}(\mu)$, and the polynomial $\phi_n^{(\alpha, \beta)}(x)$ interlaces $\phi_{n-2}^{(\alpha, \beta)}(x)$.*

Proof. According to Corollary 3, we have

$$\begin{aligned} \mathbf{zr}_i(\phi_{n-2}^{(\alpha+1, \beta+1)}) &< \mathbf{zr}_i(\phi_{n-1}^{(\alpha+1, \beta+1)}) < \mathbf{zr}_{i-1}(\phi_{n-2}^{(\alpha+1, \beta+1)}), \\ \mathbf{zr}_i(\phi_{n-1}^{(\alpha+1, \beta+1)}) &< \mathbf{zr}_i(\phi_n^{(\alpha+1, \beta+1)}) < \mathbf{zr}_{i-1}(\phi_{n-1}^{(\alpha+1, \beta+1)}) \end{aligned} \quad (33)$$

for any natural $i \leq n/2$. From Corollary 9 we obtain that

$$\mathbf{zr}_i(\phi_{n-2}^{(\alpha+1, \beta+1)}) < \mathbf{zr}_i(\phi_n^{(\alpha, \beta)}) < \mathbf{zr}_{i-1}(\phi_{n-2}^{(\alpha+1, \beta+1)}), \quad (34)$$

$$\mathbf{zr}_i(\phi_n^{(\alpha+1, \beta+1)}) < \mathbf{zr}_i(\phi_n^{(\alpha, \beta)}) < \mathbf{zr}_{i-1}(\phi_n^{(\alpha+1, \beta+1)}). \quad (35)$$

Bringing together the right inequality in (34) and the left inequalities in (35) and (33), we obtain

$$\mathbf{zr}_i(\phi_{n-2}^{(\alpha+1, \beta+1)}) \stackrel{(33)}{<} \mathbf{zr}_i(\phi_{n-1}^{(\alpha+1, \beta+1)}) \stackrel{(33)}{<} \mathbf{zr}_i(\phi_n^{(\alpha+1, \beta+1)}) \stackrel{(35)}{<} \mathbf{zr}_i(\phi_n^{(\alpha, \beta)}) \stackrel{(34)}{<} \mathbf{zr}_{i-1}(\phi_{n-2}^{(\alpha+1, \beta+1)}) \quad (36)$$

for all natural $i \leq n/2$. This relation implies that the zeros of $\phi_n^{(\alpha+1, \beta+1)}(\mu)$ and $\phi_{n-2}^{(\alpha+1, \beta+1)}(\mu)$ interlace.

By Corollary 3 we obtain (cf. (33))

$$\mathbf{zr}_i \left(\phi_{n-2}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_{n-1}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_n^{(\alpha, \beta)} \right), \quad i = 1, 2, \dots$$

This chain can be continued with the left inequality in (35) and the right inequality in (34) so that

$$\mathbf{zr}_i \left(\phi_{n-2}^{(\alpha+1, \beta+1)}(\mu) \right) \stackrel{(35)}{<} \mathbf{zr}_i \left(\phi_{n-2}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_{n-1}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_n^{(\alpha, \beta)} \right) \stackrel{(34)}{<} \mathbf{zr}_{i-1} \left(\phi_{n-2}^{(\alpha+1, \beta+1)} \right)$$

for each natural i . □

5 Conclusion: relations between Conjecture A and Conjecture B

Here we obtain two instructive facts giving an idea about the limits of the current approach. In the present note, we used a modification of the method applied in [3], so it has the same deficiency: the parameters α and β are constrained to provide the positivity of $w'_{\alpha, \beta}(x)$ and the convergence of integrals. However, these sufficient conditions seem to be quite far from being necessary.

The following two lemmata coupled with Theorem 10 give no new parameter range for Conjecture A to hold. At the same time, the comparison to Lemma 2 clearly shows that this conjecture is less restrictive than Conjecture B.

Lemma 11. *If the polynomials $\phi_n^{(\alpha, \beta)}(\mu)$, $\phi_{n-1}^{(\alpha, \beta)}(\mu)$ and $\phi_{n-2}^{(\alpha, \beta)}(\mu)$ are pairwise interlacing in such a way that*

$$\mathbf{zr}_1 \left(\phi_{n-2}^{(\alpha, \beta)} \right) < \mathbf{zr}_1 \left(\phi_{n-1}^{(\alpha, \beta)} \right) < \mathbf{zr}_1 \left(\phi_n^{(\alpha, \beta)} \right),$$

then $\phi_n^{(\alpha, \beta-1)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha, \beta-1)}(\mu)$.

Proof. By the formula (4), $\phi_n^{(\alpha, \beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha, \beta-1)}(\mu)$ have only real zeros. Furthermore,

$$\mathbf{zr}_{i+1} \left(\phi_n^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_{n-2}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_{n-1}^{(\alpha, \beta-1)} \right) < \mathbf{zr}_i \left(\phi_{n-1}^{(\alpha, \beta)} \right) < \mathbf{zr}_i \left(\phi_n^{(\alpha, \beta-1)} \right) < \mathbf{zr}_i \left(\phi_n^{(\alpha, \beta)} \right)$$

for natural $i = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$. This implies the interlacing property for the polynomials $\phi_n^{(\alpha, \beta-1)}(\mu)$ and $\phi_{n-1}^{(\alpha, \beta-1)}(\mu)$. □

As an intermediate result (Corollary 9) we had the interlacing property of $\phi_n^{(\alpha, \beta)}(\mu)$ and $\phi_n^{(\alpha+1, \beta+1)}(\mu)$ when the parameters satisfy $-1 < \alpha < 0 < \beta$. Such a fact allows us to get a relationship complementing Lemma 11.

Lemma 12. *Let the polynomial pairs $\left(\phi_n^{(\alpha, \beta)}(\mu), \phi_n^{(\alpha+1, \beta+1)}(\mu) \right)$, $\left(\phi_n^{(\alpha, \beta)}(\mu), \phi_{n-1}^{(\alpha+1, \beta+1)}(\mu) \right)$ be interlacing in such a way that*

$$\mathbf{zr}_1 \left(\phi_{n-1}^{(\alpha+1, \beta+1)} \right) < \mathbf{zr}_1 \left(\phi_n^{(\alpha, \beta)} \right), \quad \mathbf{zr}_1 \left(\phi_n^{(\alpha+1, \beta+1)} \right) < \mathbf{zr}_1 \left(\phi_n^{(\alpha, \beta)} \right). \quad (37)$$

Then $\phi_n^{(\alpha+1, \beta)}(\mu)$ interlaces $\phi_{n-1}^{(\alpha+1, \beta)}(\mu)$.

Proof. The identities (4) and (5) give

$$(n + \alpha + \beta + 2) \phi_n^{(\alpha+1, \beta+1)} + (n + \alpha + 1) \phi_{n-1}^{(\alpha+1, \beta+1)} = (2n + \alpha + \beta + 2) \phi_n^{(\alpha+1, \beta)}, \quad (38)$$

$$(n + \alpha + \beta + 1) \phi_n^{(\alpha+1, \beta)} - (2n + \alpha + \beta + 1) \phi_n^{(\alpha, \beta)} = (n + \beta) \phi_{n-1}^{(\alpha+1, \beta)}. \quad (39)$$

From the inequalities (37) we obtain that each interval $\left(\mathbf{zr}_{i+1} \left(\phi_n^{(\alpha, \beta)} \right), \mathbf{zr}_i \left(\phi_n^{(\alpha, \beta)} \right) \right)$, $i = 1, \dots, \lfloor n/2 \rfloor - 1$, contains the points $\mathbf{zr}_i \left(\phi_{n-1}^{(\alpha+1, \beta+1)} \right)$ and $\mathbf{zr}_i \left(\phi_n^{(\alpha+1, \beta+1)} \right)$ and no other zeros of the polynomials $\phi_{n-1}^{(\alpha+1, \beta+1)}$ and $\phi_n^{(\alpha+1, \beta+1)}$. Thus, the left-hand side of (38) also has exactly one zero on each of the intervals. As a result, $\phi_n^{(\alpha+1, \beta)}$ interlaces $\phi_n^{(\alpha, \beta)}$, so the zeros of their difference appearing in (39) and hence of $\phi_{n-1}^{(\alpha+1, \beta)}$ are interlacing with the zeros of $\phi_n^{(\alpha+1, \beta)}$. □

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